

Established pseudo solution of second-order Dirac-Coulomb equation with position-dependent mass

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We show that one of exact solutions of the second-order Dirac-Coulomb equation were pseudo. In the corresponding original literature, it was considered that the mass of the electron with a Coulomb potential was position-dependent, but the obtained eigenvalues set was not the inevitable mathematical deduction of the given second-order Dirac equation, and the second-order Dirac equations were not the inevitable mathematical deduction of the given couplet first-order Dirac equation with the position-dependent mass of the electron. In the present paper, we obtain the correct solution of the introduced first-order differential equations. This new solution would be tenable only when the wave equation is correct, but there is not any experiment date to validate the so-called position-dependent of the electron in the Coulomb field.

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I. INTRODUCTION

An exact solution of the Dirac equation with the so-called position-dependent mass of the electron in the Coulomb field was introduced[1]. According to the article, in atomic units ($\hbar = m_0 = 1$), the spherically symmetric singular mass distribution was taken as follows

$$m(r) = 1 + \mu\lambda^2/r \quad (1)$$

where λ is the Compton wavelength $\hbar/m_0c = c^{-1}$, and μ is a real scale parameter with inverse length dimension. The author presented some reasons why he introduced the position-dependent mass of the electron. We find the reasons were independent of any physical logic. For example, it was of that the rest mass of the particle ($m_0 = 1$) was obtained either as the asymptotic limit ($r \rightarrow \infty$), or the nonrelativistic limit ($\lambda \rightarrow 0$) of $m(r)$, consequently, a possible interpretation for this singular mass term might be found in relativistic quantum field theory. It was even told of that it should also be noted that this position-dependent mass term has a relativistic origin as well since it was proportional to the Compton wavelength which vanishes as $c \rightarrow \infty$ (equivalently, $\lambda \rightarrow 0$).

Whereas we firstly query a question here, did any experiment ever show that the mass of electron in the Coulomb field is relative to the position? One can also find that many such suppositions in the corresponding published paper are very vexed. We are clear that there is not any consequence for the m_0 and relativistic result. Of course, at present, we have to transitorily avoid such questions and only check the corresponding mathematic deduction procedure. We show that the original solution of the second-order Dirac equation with the so-called position-dependent mass was incorrect, and the given second-order Dirac equation is not the necessary mathematical deduction of the given first-order Dirac equation with position-dependent mass of the electron in the Coulomb field. We introduce the correct exact solution of the original first-order differential equation only for further showing that the original solution includes many mathematical mistakes, and don't think the supposition of the position-dependent mass of the electron in the Coulomb field is correct.

II. ESTABLISHED SOLUTION OF DIRAC EQUATION WITH POSITION-DEPENDENT MASS

In order to solve the Dirac equation with the so-called position-dependent mass term in the Coulomb field, the spinor wavefunction was written as follows

$$\psi = \begin{pmatrix} i[g(r)/r] \chi_{lm}^j \\ [f(r)/r] \vec{\sigma} \cdot \chi_{lm}^j \end{pmatrix} \quad (2)$$

where f and g are real radial functions, \hat{r} is the radial unit vector, and the angular wavefunction with the spherical harmonic function $Y_l^{m-1/2}$ for the two-component spinor was written as

$$\chi_{lm}^j = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l \pm m + 1/2} & Y_l^{m-1/2} \\ \mp \sqrt{l \mp m + 1/2} & Y_l^{m+1/2} \end{pmatrix} \quad (3)$$

then the following 2×2 matrix equation for the two radial spinor components was given

$$\begin{pmatrix} 1 + \lambda^2 \frac{Z+\mu}{r} - \varepsilon & \lambda \left(\frac{k}{r} - \frac{d}{dr} \right) \\ \lambda \left(\frac{k}{r} + \frac{d}{dr} \right) & -1 + \lambda^2 \frac{Z-\mu}{r} - \varepsilon \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0 \quad (4)$$

where ε is the relativistic energy which is real. By introducing some signs and using the global unitary transformation, it was alleged that, the Schrodinger-like wave equation was finally obtained. The main steps are as follows. The Schrodinger-like requirement dictates that the parameter η satisfies the constraint

$$C\mu + S\kappa/\lambda = \pm Z \quad (5)$$

where $S = \sin(\lambda\eta)$, $C = \cos(\lambda\eta)$ and $-\pi/2 \leq \lambda\eta \leq \pi/2$, $\kappa = \pm 1, \pm 2, \dots$. The solution of the constraint (5) gives two angles whose cosines are

$$C = (\mu^2 + \kappa^2/\lambda^2)^{-1} \left[\pm \mu Z + \frac{|\kappa|}{\lambda} \sqrt{\left(\frac{\kappa}{\lambda}\right)^2 + \mu^2 - Z^2} \right] > 0 \quad (6)$$

the equation (4) is now transformed into the following

$$\begin{pmatrix} C - \varepsilon + (1 \pm 1) \lambda^2 \frac{Z}{r} & \lambda \left(-\frac{s}{\lambda} + \frac{\gamma}{r} - \frac{d}{dr} \right) \\ \lambda \left(-\frac{s}{\lambda} + \frac{\gamma}{r} + \frac{d}{dr} \right) & -C - \varepsilon + (1 \mp 1) \lambda^2 \frac{Z}{r} \end{pmatrix} \begin{pmatrix} \phi^+(r) \\ \phi^-(r) \end{pmatrix} = 0 \quad (7)$$

where $\gamma = \frac{|\kappa|}{\kappa} \sqrt{\kappa^2 + \lambda^2(\mu^2 - Z^2)}$ and

$$\begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = \begin{pmatrix} \cos \frac{\lambda\eta}{2} & \sin \frac{\lambda\eta}{2} \\ -\sin \frac{\lambda\eta}{2} & \cos \frac{\lambda\eta}{2} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} \quad (8)$$

equation (7) gives one spinor component in terms of the other as follows

$$\phi^\pm = \frac{\lambda}{C \pm \varepsilon} \left(\pm \frac{S}{\lambda} \mp \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^\mp \quad (9)$$

whereas, the resulting Schrödinger-like wave equation becomes

$$\left[-\frac{d^2}{dr^2} + \frac{\gamma(\gamma \pm 1)}{r^2} + 2\frac{Z\varepsilon + \mu}{r} - \frac{\varepsilon^2 - 1}{\lambda^2} \right] \phi^\mp(r) = 0 \quad (10)$$

the author comparing this equation with that of the well-known nonrelativistic Coulomb problem with constant mass and alleged that he had new discover for the relativistic spectrum

$$\varepsilon_n^\iota = \left[1 + \left(\frac{\lambda Z}{n + \iota + 1} \right)^2 \right]^{-1} \left[-\frac{\lambda^2 \mu Z}{(n + \iota + 1)^2} \pm \sqrt{1 + \lambda^2 \frac{Z^2 - \mu^2}{(n + \iota + 1)^2}} \right] \quad (11)$$

namely

$$\varepsilon_n^\iota = \frac{-\lambda^2 \mu Z \pm \sqrt{(n + \iota + 1)^4 + \lambda^2 (Z^2 - \mu^2) (n + \iota + 1)^2}}{(n + \iota + 1)^2 + (\lambda Z)^2} \quad (12)$$

where ι stands for either one of the four possible alternative values in the following expression associated independently with ϕ^\pm and $\pm\kappa \geq 1$. For ϕ^+

$$\iota \rightarrow \gamma \text{ or } \iota \rightarrow -\gamma - 1, \quad Z \rightarrow Z\varepsilon + \mu, \quad E \rightarrow (\varepsilon^2 - 1)/2\lambda^2 \quad (13)$$

and for ϕ^-

$$\iota \rightarrow \gamma - 1 \text{ or } \iota \rightarrow -\gamma, \quad Z \rightarrow Z\varepsilon + \mu, \quad E \rightarrow (\varepsilon^2 - 1)/2\lambda^2 \quad (14)$$

It is similar to the above procedure, writing the paper about the relativistic quantum mechanics, many authors did not introduce their detail operation steps on how to obtain those necessary transition equation and their new mathematical result. They only alleged that those formulas in their paper are necessary deduction. We don't understand what the expression (9) means?

III. EINGEVALUES-SET (12) DISOBEY UNIQUENESS OF SOLUTION OF WAVE EQUATION

For the same quantum system, it should have only one of eigenvalues set for any theory. The formula (12) of the energy levels includes two eigenvalues set corresponding to different definition. It is one of the mathematical contradictions of the articles. Consequently we cannot believe the formula (12) is the real energy eigenvalues set in the Coulomb field. It seems that the author is not up on the method of finding the eigensolutions of the second-order differential equations with variable coefficients. Why don't we directly solve the second-order differential equation (10) now? Now, one can write the equation (10) in the separate form

$$\begin{cases} \left[-\frac{d^2}{dr^2} + \frac{\gamma(\gamma+1)}{r^2} + 2\frac{Z\varepsilon+\mu}{r} - \frac{\varepsilon^2-1}{\lambda^2} \right] \phi(r) = 0 \\ \left[-\frac{d^2}{dr^2} + \frac{\gamma(\gamma-1)}{r^2} + 2\frac{Z\varepsilon+\mu}{r} - \frac{\varepsilon^2-1}{\lambda^2} \right] \psi(r) = 0 \end{cases} \quad (15)$$

using the asymptotic solution $\phi(r) \sim \exp(-r\sqrt{1-\varepsilon^2}/\lambda)$, $\psi(r) \sim \exp(-r\sqrt{1-\varepsilon^2}/\lambda)$ satisfying the boundary condition at $r \rightarrow 0$, we seek the formal solution

$$\phi(r) = e^{-\frac{\sqrt{1-\varepsilon_u^2}}{\lambda}r} u, \quad \psi(r) = e^{-\frac{\sqrt{1-\varepsilon_v^2}}{\lambda}r} v \quad (16)$$

it easily obtained that

$$\begin{aligned} \frac{d\phi(r)}{dr} &= e^{-\frac{\sqrt{1-\varepsilon_u^2}}{\lambda}r} \left(\frac{du}{dr} - \frac{\sqrt{1-\varepsilon_u^2}}{\lambda} u \right) \\ \frac{d^2\phi(r)}{dr^2} &= e^{-\frac{\sqrt{1-\varepsilon_u^2}}{\lambda}r} \left(\frac{d^2u}{dr^2} - \frac{2\sqrt{1-\varepsilon_u^2}}{\lambda} \frac{du}{dr} + \frac{1-\varepsilon_u^2}{\lambda^2} u \right) \\ \frac{d\psi(r)}{dr} &= e^{-\frac{\sqrt{1-\varepsilon_v^2}}{\lambda}r} \left(\frac{dv}{dr} - \frac{\sqrt{1-\varepsilon_v^2}}{\lambda} v \right) \\ \frac{d^2\psi(r)}{dr^2} &= e^{-\frac{\sqrt{1-\varepsilon_v^2}}{\lambda}r} \left(\frac{d^2v}{dr^2} - \frac{2\sqrt{1-\varepsilon_v^2}}{\lambda} \frac{dv}{dr} + \frac{1-\varepsilon_v^2}{\lambda^2} v \right) \end{aligned} \quad (17)$$

substituting (16) and (17) into (15), it educes that

$$\begin{aligned} \frac{d^2u}{dr^2} - \frac{2\sqrt{1-\varepsilon_u^2}}{\lambda} \frac{du}{dr} - \frac{\gamma(\gamma+1)}{r^2} u - 2\frac{Z\varepsilon_u+\mu}{r} u &= 0 \\ \frac{d^2v}{dr^2} - \frac{2\sqrt{1-\varepsilon_v^2}}{\lambda} \frac{dv}{dr} - \frac{\gamma(\gamma-1)}{r^2} v - 2\frac{Z\varepsilon_v+\mu}{r} v &= 0 \end{aligned} \quad (18)$$

finding the power series solution of the above equations, it assumed that

$$u = \sum_{n=0}^{\infty} b_n r^{s_u+n}, \quad v = \sum_{n=0}^{\infty} d_n r^{s_v+n} \quad (19)$$

so that

$$\begin{aligned}
\frac{du}{dr} &= \sum_{n=0}^{\infty} (s_u + n) b_n r^{s_u+n-1} \\
\frac{dv}{dr} &= \sum_{n=0}^{\infty} (s_v + n) d_n r^{s_v+n-1} \\
\frac{d^2u}{dr^2} &= \sum_{n=0}^{\infty} (s_u + n) (s_u + n - 1) b_n r^{s_u+n-2} \\
\frac{d^2v}{dr^2} &= \sum_{n=0}^{\infty} (s_v + n) (s_v + n - 1) d_n r^{s_v+n-2}
\end{aligned} \tag{20}$$

substituting (19) and (20) into the equations (17), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left\{ \begin{aligned} &[(s_u + n) (s_u + n - 1) - \gamma (\gamma + 1)] b_n \\ &- 2 \left[\frac{\sqrt{1-\varepsilon_u^2}}{\lambda} (s_u + n - 1) + (Z\varepsilon_u + \mu) \right] b_{n-1} \end{aligned} \right\} r^{s_u+n-2} &= 0 \\
\sum_{n=0}^{\infty} \left\{ \begin{aligned} &[(s_v + n) (s_v + n - 1) - \gamma (\gamma - 1)] d_n \\ &- 2 \left[\frac{\sqrt{1-\varepsilon_v^2}}{\lambda} (s_v + n - 1) + (Z\varepsilon_v + \mu) \right] d_{n-1} \end{aligned} \right\} r^{s_v+n-2} &= 0
\end{aligned} \tag{21}$$

finally we obtain the two recursive relation for the power series (19)

$$\begin{aligned}
[(s_u + n) (s_u + n - 1) - \gamma (\gamma + 1)] b_n - 2 \left[\frac{\sqrt{1-\varepsilon_u^2}}{\lambda} (s_u + n - 1) + (Z\varepsilon_u + \mu) \right] b_{n-1} &= 0 \\
[(s_v + n) (s_v + n - 1) - \gamma (\gamma - 1)] d_n - 2 \left[\frac{\sqrt{1-\varepsilon_v^2}}{\lambda} (s_v + n - 1) + (Z\varepsilon_v + \mu) \right] d_{n-1} &= 0
\end{aligned} \tag{22}$$

using the initial value condition $b_{-1} = b_{-2} = \dots = 0$, $d_{-1} = d_{-2} = \dots = 0$ and $b_0 \neq 0$, $d_0 \neq 0$, let $n = 0$ and substitute it into the above recursive relation, it educes that

$$\begin{aligned}
s_u (s_u - 1) - \gamma (\gamma + 1) &= 0 \\
s_v (s_v - 1) - \gamma (\gamma - 1) &= 0
\end{aligned} \tag{23}$$

this gives

$$s_{u1} = 1 + \gamma, \quad s_{u2} = -\gamma; \quad s_{v1} = \gamma, \quad s_{v2} = 1 - \gamma \tag{24}$$

in order to that the whole solutions (2) of the original equation satisfy the boundary condition, we have to choose

$$s_{u1} = 1 + \gamma, \quad s_{v1} = \gamma \tag{25}$$

hence

$$u = \sum_{n=0}^{\infty} b_n r^{1+\gamma+n}, \quad v = \sum_{n=0}^{\infty} d_n r^{\gamma+n} \tag{26}$$

and

$$\phi(r) = e^{-\frac{\sqrt{1-\varepsilon_u^2}}{\lambda} r} \sum_{n=0}^{\infty} b_n r^{1+\gamma+n}, \quad \psi(r) = e^{-\frac{\sqrt{1-\varepsilon_v^2}}{\lambda} r} \sum_{n=0}^{\infty} d_n r^{\gamma+n} \tag{27}$$

the power series must be cut off so that the whole wave function is limit at $r \rightarrow \infty$. It is assumed that $b_{n_r} \neq 0$, $d_{n_r} \neq 0$ and $b_{n_r+1} = b_{n_r+2} = \dots = 0$, $d_{n_r+1} = d_{n_r+2} = \dots = 0$. According to the recursive relation (22), let $n = n_r + 1$, we obtain

$$\begin{aligned} - \left[\frac{\sqrt{1-\varepsilon_u^2}}{\lambda} (s_u + n_r) + (Z\varepsilon_u + \mu) \right] b_{n_r} &= 0 \\ - \left[\frac{\sqrt{1-\varepsilon_v^2}}{\lambda} (s_v + n_r) + (Z\varepsilon_v + \mu) \right] d_{n_r} &= 0 \end{aligned} \quad (28)$$

it requires that

$$\begin{aligned} \frac{\sqrt{1-\varepsilon_u^2}}{\lambda} (1 + \gamma + n_r) + (Z\varepsilon_u + \mu) &= 0 \\ \frac{\sqrt{1-\varepsilon_v^2}}{\lambda} (\gamma + n_r) + (Z\varepsilon_v + \mu) &= 0 \end{aligned} \quad (29)$$

if we think little of, we would obtain the formal solution

$$\begin{aligned} \varepsilon_{u1} &= \frac{-\lambda^2 Z\mu + \sqrt{(1+\gamma+n_r)^4 + \lambda^2 Z^2 [\lambda^2 \mu^2 + (1+\gamma+n_r)^2]}}{\lambda^2 Z^2 + (1+\gamma+n_r)^2} \\ \varepsilon_{u2} &= \frac{-\lambda^2 Z\mu - \sqrt{(1+\gamma+n_r)^4 + \lambda^2 Z^2 [\lambda^2 \mu^2 + (1+\gamma+n_r)^2]}}{\lambda^2 Z^2 + (1+\gamma+n_r)^2} \\ \varepsilon_{v1} &= \frac{-\lambda^2 Z\mu + \sqrt{(\gamma+n_r)^4 + \lambda^2 Z^2 [\lambda^2 \mu^2 + (\gamma+n_r)^2]}}{\lambda^2 Z^2 + (\gamma+n_r)^2} \\ \varepsilon_{v2} &= \frac{-\lambda^2 Z\mu - \sqrt{(\gamma+n_r)^4 + \lambda^2 Z^2 [\lambda^2 \mu^2 + (\gamma+n_r)^2]}}{\lambda^2 Z^2 + (\gamma+n_r)^2} \end{aligned} \quad (30)$$

in form, these results as the inevitable deductions of the second-order differential equations (14) include omnifarious logic problems.

a) The solutions (30) are different from the formula (12). It shows that the original formula (12) is incorrect for the second-order differential equations (4).

b) For the same quantum system described by the equations (14), the four eigenvalues sets of the energy levels disobey the uniqueness of the solution of the differential equations. Which eigenvalues set is correct?

c) Is the relativistic energy the positive number or the negative number? If we delete the negative energy solution, we also have two eigenvalues set corresponding the positive energy. They also disobey the uniqueness of the eigenvalues set for the differential equation.

d) It is the most serious that, the solutions (30) implying the (12) are the formal solution. Because of the definition $\gamma = \frac{|\kappa|}{\kappa} \sqrt{\kappa^2 + \lambda^2 (\mu^2 - Z^2)}$ given in the original article, only when $\kappa < 0$, the quadratic equations with one unknown can have the solutions (30). Whereas the κ values constructed by Dirac are actually $\pm 1, \pm 2, \dots$, also given in the original articles.

All of these problems are the mathematical and physical contradictions. Consequently, we don't think the original solution (12) is not deceitful solution.

IV. SCHRÖDINGER-LIKE EQUATION (10) IS BOGUS

We even doubt that the second-order equation (10) is the correct deduction of the original coupled first-order equation (4). By all appearances, from (4) to (10), it actually introduces the transformation (8) with (9), namely

$$\begin{pmatrix} \frac{\lambda}{C+\varepsilon} \left(\frac{S}{\lambda} - \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^- \\ \frac{\lambda}{C-\varepsilon} \left(-\frac{S}{\lambda} + \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^+ \end{pmatrix} = \begin{pmatrix} \cos \frac{\lambda\eta}{2} & \sin \frac{\lambda\eta}{2} \\ -\sin \frac{\lambda\eta}{2} & \cos \frac{\lambda\eta}{2} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} \quad (31)$$

it gives the separate form

$$\begin{aligned} g &= \frac{\frac{\lambda}{C+\varepsilon} \left(\frac{S}{\lambda} - \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^- - \frac{\lambda}{C-\varepsilon} \left(-\frac{S}{\lambda} + \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^+}{\cos \frac{\lambda\eta}{2} + \sin \frac{\lambda\eta}{2}} \\ f &= \frac{\frac{\lambda}{C+\varepsilon} \left(\frac{S}{\lambda} - \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^- + \frac{\lambda}{C-\varepsilon} \left(-\frac{S}{\lambda} + \frac{\gamma}{r} + \frac{d}{dr} \right) \phi^+}{\cos \frac{\lambda\eta}{2} + \sin \frac{\lambda\eta}{2}} \end{aligned} \quad (32)$$

these transformations cannot translate the equations (4) namely

$$\begin{aligned} \left(1 + \lambda^2 \frac{Z+\mu}{r} - \varepsilon \right) g(r) + \lambda \left(\frac{k}{r} - \frac{d}{dr} \right) f(r) &= 0 \\ \lambda \left(\frac{k}{r} + \frac{d}{dr} \right) g(r) - \left(-1 + \lambda^2 \frac{Z-\mu}{r} - \varepsilon \right) f(r) &= 0 \end{aligned} \quad (33)$$

into any Schrödinger-like second-order equations, and the second-order equation (10) give in the original paper is not correct.

V. CORRECT SOLUTION OF THE CORRESPONDING FIRST-ORDER EQUATION

Because there has been not any experiment data to approve the imagination, we don't think that the mass of electron in the Coulomb field is really dependent on position. Consequently, in principle, it should be meaningless to find the correct solution of the equation (4) or (33). When only looking from a mathematical point of view, we give the correct exact solution of the equations (4) or (33). It is well known that directly solving the original coupled first-order equation (4) namely (33) is simpler out and away than translating it into the so-called Schrödinger-like equation to obtain the exact solution. One firstly note the behavior of $f(r)$ and $g(r)$ for $g \rightarrow \infty$, since neglecting the terms proportional to $1/r$ the differential equations (33) read

$$(1 - \varepsilon) g - \lambda \frac{df}{dr} = 0, \quad \lambda \frac{dg}{dr} + (1 + \varepsilon) f = 0 \quad (34)$$

it follows immediately that

$$\frac{d^2 f}{dr^2} + \frac{1 - \varepsilon^2}{\lambda^2} f \sim 0, \quad \frac{d^2 g}{dr^2} + \frac{1 - \varepsilon^2}{\lambda^2} g \sim 0 \quad (35)$$

there will be different solution with the different fields of definitions of the ε . However, it is considered all along that the relativistic energy in the Coulomb field satisfy the condition $0 < E < m_0 c^2$ namely $E < 1$. According to (13) and (14), it only gives $1 < \varepsilon^2 < 2\lambda^2 + 1$. For the moment, it is considered that $\varepsilon^2 > 1$. We obtain the asymptotic solutions of the equations (33)

$$f \sim e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r}, \quad g \sim e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} \quad (36)$$

and the exact solution of the equations (33) take form

$$f = e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r}u, \quad g = e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r}v \quad (37)$$

they give

$$\begin{aligned} \frac{df}{dr} &= e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} \frac{du}{dr} - \frac{\sqrt{\varepsilon^2-1}}{\lambda} e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} u \\ \frac{dg}{dr} &= e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} \frac{dv}{dr} - \frac{\sqrt{\varepsilon^2-1}}{\lambda} e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} v \end{aligned} \quad (38)$$

substituting for equations (33), we have

$$\begin{aligned} \lambda \frac{du}{dr} - \left(\frac{\lambda k}{r} + \sqrt{\varepsilon^2 - 1} \right) u + \left[(\varepsilon - 1) - \frac{\lambda^2(Z+\mu)}{r} \right] v &= 0 \\ \lambda \frac{dv}{dr} + \left(\frac{\lambda k}{r} - \sqrt{\varepsilon^2 - 1} \right) v + \left[(\varepsilon + 1) - \frac{\lambda^2(Z-\mu)}{r} \right] u &= 0 \end{aligned} \quad (39)$$

finding the power series solution, put

$$v = \sum_{n=0}^{\infty} b_n r^{\sigma+n}, \quad u = \sum_{n=0}^{\infty} d_n r^{\sigma+n} \quad (40)$$

substitute it into the above equations, we obtain

$$\begin{aligned} \lambda \sum_{n=0}^{\infty} (\sigma + n) d_n r^{\sigma+n-1} - \left(\frac{\lambda k}{r} + \sqrt{\varepsilon^2 - 1} \right) \sum_{n=0}^{\infty} d_n r^{\sigma+n} + \left[(\varepsilon - 1) - \frac{\lambda^2(Z+\mu)}{r} \right] \sum_{n=0}^{\infty} b_n r^{\sigma+n} &= 0 \\ \lambda \sum_{n=0}^{\infty} (\sigma + n) b_n r^{\sigma+n-1} + \left(\frac{\lambda k}{r} - \sqrt{\varepsilon^2 - 1} \right) \sum_{n=0}^{\infty} b_n r^{\sigma+n} + \left[(\varepsilon + 1) - \frac{\lambda^2(Z-\mu)}{r} \right] \sum_{n=0}^{\infty} d_n r^{\sigma+n} &= 0 \end{aligned} \quad (41)$$

it predigests that

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\lambda(\sigma + n - k) d_n - \sqrt{\varepsilon^2 - 1} d_{n-1} + (\varepsilon - 1) b_{n-1} - \lambda^2(Z + \mu) b_n \right] r^{\sigma+n-1} &= 0 \\ \sum_{n=0}^{\infty} \left[\lambda^2(Z - \mu) d_n - (\varepsilon + 1) d_{n-1} - \lambda(\sigma + n + k) b_n + \sqrt{\varepsilon^2 - 1} b_{n-1} \right] r^{\sigma+n-1} &= 0 \end{aligned} \quad (42)$$

so the coefficients of the power series satisfy the recursive relations

$$\begin{aligned} \lambda(\sigma + n - k) d_n - \sqrt{\varepsilon^2 - 1} d_{n-1} + (\varepsilon - 1) b_{n-1} - \lambda^2(Z + \mu) b_n &= 0 \\ \lambda^2(Z - \mu) d_n - (\varepsilon + 1) d_{n-1} - \lambda(\sigma + n + k) b_n + \sqrt{\varepsilon^2 - 1} b_{n-1} &= 0 \end{aligned} \quad (43)$$

Solving the above recursive relations and using the initial conditions that $b_0 \neq 0$, $b_{-1} = b_{-2} = \dots = 0$ and $d_0 = 0$, $d_{-1} = d_{-2} = \dots = 0$, put $n = 0$ in (43), we obtain

$$\begin{aligned}\lambda(\sigma - k)d_0 - \lambda^2(Z + \mu)b_0 &= 0 \\ \lambda^2(Z - \mu)d_0 - \lambda(\sigma + k)b_0 &= 0\end{aligned}\tag{44}$$

it requests that the determinant of coefficient is equivalent to zero, so that

$$\begin{vmatrix} \lambda(\sigma - k) & -\lambda^2(Z + \mu) \\ \lambda^2(Z - \mu) & -\lambda(\sigma + k) \end{vmatrix} = 0\tag{45}$$

it educes that value of the index $\sigma = \pm\sqrt{k^2 + \lambda^2(Z^2 - \mu^2)}$, since the wave function must be limited at $r \rightarrow 0$, it can be only taken

$$\sigma = \sqrt{k^2 + \lambda^2(Z^2 - \mu^2)}\tag{46}$$

on the other hand, the formal whole wave function should be combined by expressions (2), (3), (37), (40) and (46), that is to say

$$\psi = \begin{pmatrix} ie^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} \sum_{n=0}^{\infty} b_n r^{\sqrt{k^2+\lambda^2(Z^2-\mu^2)}+n-1} \chi_{lm}^j \\ e^{-\frac{\sqrt{\varepsilon^2-1}}{\lambda}r} \sum_{n=0}^{\infty} d_n r^{\sqrt{k^2+\lambda^2(Z^2-\mu^2)}+n-1} \vec{\sigma} \cdot \chi_{lm}^j \end{pmatrix}\tag{47}$$

the boundary condition at $r \rightarrow \infty$ requests that the parts of the power series in the formal whole wave function must be cut off. It is assumed that $b_{n_r} \neq 0$, $d_{n_r} \neq 0$ and $b_{n_r+1} = b_{n_r+2} = \dots = 0$, $d_{n_r+1} = d_{n_r+2} = \dots = 0$, according to the recursive relations (43), put $n = n_r + 1$, we obtain

$$\begin{aligned}-\sqrt{\varepsilon^2-1}d_{n_r} + (\varepsilon-1)b_{n_r} &= 0 \\ -(\varepsilon+1)d_{n_r} + \sqrt{\varepsilon^2-1}b_{n_r} &= 0\end{aligned}\tag{48}$$

it indicates that the ε should take some special values so that $b_{n_r+1} = 0$ and $d_{n_r+1} = 0$. Note that the recursive relations (43). Multiplying the first relation by $\sqrt{\varepsilon^2-1}$ and multiplying the second relation by $\varepsilon-1$, we have

$$\begin{aligned}\lambda(\sigma + n_r - k)\sqrt{\varepsilon^2-1}d_{n_r} - (\varepsilon^2-1)d_{n_r-1} \\ -\lambda^2(Z + \mu)\sqrt{\varepsilon^2-1}b_{n_r} + (\varepsilon-1)\sqrt{\varepsilon^2-1}b_{n_r-1} &= 0 \\ \lambda^2(Z - \mu)(\varepsilon-1)d_{n_r} - (\varepsilon^2-1)d_{n_r-1} \\ -\lambda(\sigma + n_r + k)(\varepsilon-1)b_{n_r} + (\varepsilon-1)\sqrt{\varepsilon^2-1}b_{n_r-1} &= 0\end{aligned}\tag{49}$$

this deduces that

$$d_{n_r} = -\frac{(\sigma + n_r + k)(\varepsilon-1) - \lambda(Z + \mu)\sqrt{\varepsilon^2-1}}{(\sigma + n_r - k)\sqrt{\varepsilon^2-1} - \lambda(Z - \mu)(\varepsilon-1)}b_{n_r}\tag{50}$$

combing this relations and one of the system of the recursive relations (48), we obtain

$$(\sigma + n_r) \sqrt{\varepsilon^2 - 1} = \lambda Z \varepsilon + \lambda \mu \quad (51)$$

it gives that

$$\varepsilon = \frac{\lambda^2 \mu Z \pm (\sigma + n_r) \sqrt{(\sigma + n_r)^2 + \lambda^2 (\mu^2 - Z^2)}}{(\sigma + n_r)^2 - \lambda^2 Z^2} \quad (52)$$

we find that it is different from not only the formula (11) given in the original articles but also the correct formula (30) of the second-order differential equations. This shows that the original exact solution of the Dirac equation with position-dependent mass is the pseudo solution.

VI. CONCLUSIONS

We have used the basal knowledge of the differential equation with variable coefficient to show that, almost every step of the mathematical operation in the mentioned paper for introducing the Dirac theory with the position-dependent mass of the electron in the Coulomb field is incorrect, and the energy eigenvalues so the eigen-wave-function given in the original paper are pseudo. Only in a mathematical signification, we give the correct solution and eigenvalue set of the first-order differential equations. However we don't regard the formula (52) of the energy-levels as the necessary result of the development of quantum mechanics. Because the position-dependent mass of the electron in the Coulomb field make us discredit its authenticity.

Some other papers also alleged that they found the exact solution of the Dirac equation for a particle with position-dependent mass[2], which might be useful in the study of the corresponding non-relativistic problem as a reference result. It was even considered that the next terms, which they have neglected in this work (in particular the dipolar one) and which are responsible for the super-fine structure of the energy spectrum, can be taken into account by means of standard perturbation theory. Although the corresponding paper constructed the second-order Dirac equation by using unconvencionality methods which also can be seen in the some papers published 25 years ago[3]. However, is this theory correct? Without solving the corresponding differential equation, one can find some methods to directly conclude that some other papers about the Dirac theory are not correct.

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